Stokes' Theorem

$F: \mathbb{R}^3 \to \mathbb{R}^3$ vector field

S surface in $\mathbb{R}^3$, bounded by curve C

$$\iint_S \operatorname{curl} F \cdot dS = \int_C F \cdot ds$$

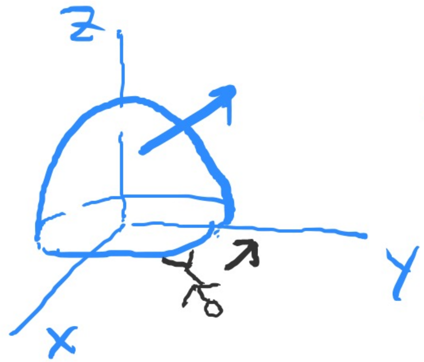
with compatible orientations

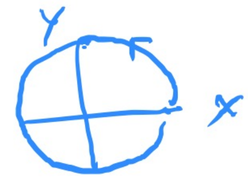
Example:

Let $F(x,y,z) = (y, -x, e^{xz^2})$

and let S be the upper semisphere with radius 1

positive side = upper side.





Calculate $\iint_S \text{curl}\, F \cdot dS$

surface integral of curl of F

$\Rightarrow$ try Stokes' theorem!

here: boundary of S = unit circle in xy plane.

compatible param: $\boxed{c(t) = (\cos t, \sin t, 0)}$ $\boxed{c'(t) = (-\sin t, \cos t, 0)}$

$$\Rightarrow \iint_S \operatorname{curl} F \cdot dS = \int_0^{2\pi} F(\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$

$F(x,y,z) = (-y, x,$ compl. mess$)$

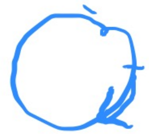
$$= \int_0^{2\pi} (\sin t, -\cos t, \text{mess}) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t \; dt$$

$$= \int_0^{2\pi} -1 \; dt = \boxed{-2\pi}$$

Question: how to parametrize clockwise orientation

$$c(t) = (\cos t, -\sin t)$$



(lazy option: use counterclockwise parametrization and put a minus sign in front of integral.)

8.3 Conservative Vector Fields

Recall: If vector field F is a gradient field

$F = \nabla f$

$\Longrightarrow$ Line integrals easy to calculate

i.e. if $c : [a,b] \to \mathbb{R}^3$

$$\Longrightarrow \int_c F \cdot ds = f(c(b)) - f(c(a))$$

Question: ① how to detect gradient fields?

② how to calculate f s.t. $F = \nabla f$

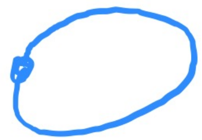
Theorem

$F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

(a) $F = \nabla f$ is a gradient field

$\Longleftrightarrow$ (b) $\text{curl } F = 0$

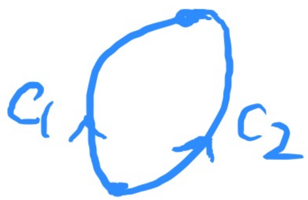
$\Longleftrightarrow$ (c) $\int_C F \cdot ds = 0$ for any closed curve C



(endpoint = initial point)

$\Longleftrightarrow$ (d) If C_1 and C_2 are curves with same initial and end points

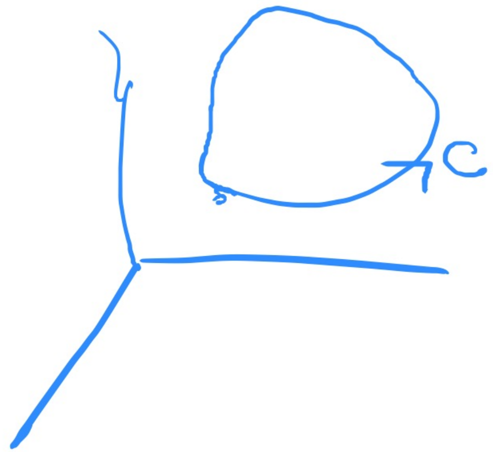
$$\Rightarrow \int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$$



justification for (b) $\Rightarrow$ (c)

assume curl $F = 0$

let C be a closed curve



Find a surface S whose boundary is C

$$\Rightarrow \int_C F \cdot ds = \iint_S \underbrace{\text{curl}\, F}_{=0} \cdot dS = 0$$

Application

Question: Is $F(x,y,z) = (y\cos xy,\ x\cos xy + z^2,\ 2yz)$ a gradient field?

Sol. curl $F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos xy & x\cos xy + z^2 & 2yz \end{vmatrix}$

$= \vec{i}\left(\partial/\partial y\,(2yz) - \partial/\partial z\,(x\cos xy + z^2)\right)$

$-\vec{j}\left(\partial/\partial x\,(2yz) - \partial/\partial z\,(y\cos xy)\right)$

$+\vec{k}\left(\partial/\partial x\,(x\cos xy + z^2) - \partial/\partial y\,(y\cos xy)\right)$

$= \vec{i}\,(\underbrace{2z - 2z}_{=0}) - \vec{j}\,(\underbrace{0 - 0}_{=0}) + \vec{k}\,(\underbrace{\cos xy - xy\sin xy - (\cos xy - xy\sin xy)}_{=0})$

$= \vec{0}.$

Answer: F is a gradient field.

how to find f s.t. $F = \nabla f$

Define $f(x,y,z) = \int_C F \cdot ds$

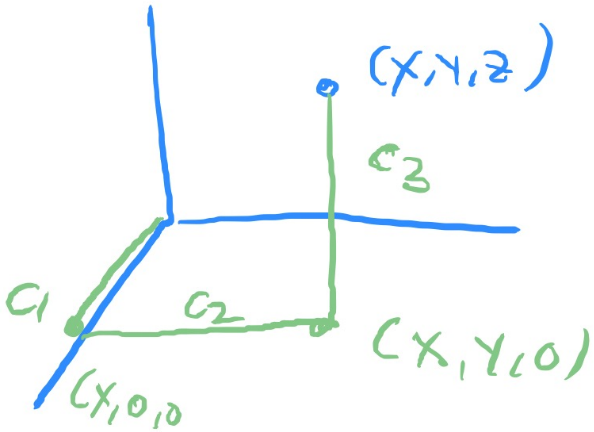
where c is any curve from 0 to (x,y,z)

By property (d) of Theorem $f(x,y,z)$ does not depend on choice of C

A convenient choice:

$C = C_1 \cup C_2 \cup C_3$, where

$$\boxed{\begin{aligned} c_1(t) &= (t,0,0) & 0 \le t \le x \\ c_2(t) &= (x,t,0) & 0 \le t \le y \\ c_3(t) &= (x,y,t) & 0 \le t \le z \end{aligned}} \Bigg\} \Rightarrow \boxed{\begin{aligned} c_1'(t) &= (1,0,0) \\ c_2'(t) &= (0,1,0) \\ c_3'(t) &= (0,0,1) \end{aligned}}$$



$$F(x,y,z) = (y\cos xy,\ x\cos xy + z^2,\ 2yz)$$

$$\Rightarrow f(x,y,z) = \int_{C_1} F\cdot ds + \int_{C_2} F\cdot ds + \int_{C_3} F\cdot dr$$

$$= \int_0^x F(t,0,0)\cdot(1,0,0)\,dt + \int_0^y F(x,t,0)\cdot(0,1,0)\,dt$$

$$+ \int_0^z F(x,y,t)\cdot(0,0,1)\,dt$$

$$= \int_0^x \underbrace{(0,t,0)\cdot(1,0,0)}_{=0}\,dt + \int_0^y (\ldots,\ x\cos tx,\ \ldots)\cdot(0,1,0)\,dt$$

$$+ \int_0^z (\ldots,\ldots,\ 2yt)\cdot(0,0,1)\,dt =$$

$$= 0 + \int_0^y x \cos tx \, dt + \int_0^z 2ty \, dt$$

$$= \sin tx \Big|_0^y + yt^2 \Big|_0^z$$

$$= \sin yx + yz^2$$

answer: $f(x,y,z) = \sin yx + yz^2$

<u>check:</u> $\nabla f = F$

2-dimensional case:

same theorem holds as in 3-dim case except statement (b) has to be replaced by

(b)' scalar curl of $F = 0$

i.e. if $F(x,y) = (P(x,y), Q(x,y))$

$$\Rightarrow \text{scalar curl} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

If $= 0$

$$\Rightarrow F = \nabla f \quad \text{for some function } f$$

where $\boxed{f(x,y) = \int_0^x F(t,0)\cdot(1,0)\,dt + \int_0^y F(x,t)\cdot(0,1)\,dt}$